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Radiation damping in the pilot wave interpretation of quantum mechanics

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Abstract. To avoid the internal contradiction in the pilot wave interpretation inherent in the assumption of a charged particle moving in a closed orbit without radiating, it is proposed that the wavefunctions, $\psi_n(\boldsymbol{r})$, corresponding to stationary states, be real, implying that the particle is stationary. If the effect of radiation damping is included, the Hamiltonian contains the nonlinear term

$$\tfrac{1}{2}\mathrm{i}\hbar\tau(\mathrm{d}^2/\mathrm{d}t^2)\ln[\Psi(\boldsymbol{r},t)/\Psi^*(\boldsymbol{r},t)],$$

resulting in the stability of the stationary states.

If we substitute $\Psi(\boldsymbol{r},t)=R\exp(\mathrm{i}S/\hbar)$, where R and S are real functions, into Schrödinger's time dependent equation for a single (charged) particle, we obtain upon separating the real and imaginary parts

$$-\partial S/\partial t=(\nabla S)^2/2M+V(\boldsymbol{r})-\hbar^2\nabla^2R/2MR \tag{1a}$$

$$\partial R^2/\partial t+\nabla\cdot(\nabla SR^2/M)=0. \tag{1b}$$

In the pilot wave interpretation (de Broglie 1927, 1964, Bohm 1952), (1a) is considered to be the quantum-mechanical generalisation of the Hamilton–Jacobi equation in which the total energy $E=-\partial S/\partial t$, the momentum $\boldsymbol{p}=\nabla S$, and the quantum potential energy $Q=-\hbar^2\nabla^2R/(2MR)$. Equation (1$b$) is then a continuity equation for $|\Psi|^2=R^2$.

If $\Psi_n(\boldsymbol{r},t)=\psi_n(\boldsymbol{r})\exp(-\mathrm{i}E_nt/\hbar)$ is a solution of Schrödinger's equation, (1a) reduces to

$$E_n=(\nabla S)^2/2M+V(\boldsymbol{r})-\hbar^2\nabla^2R/2MR. \tag{2a}$$

Furthermore, if $\psi_n(\boldsymbol{r})$ is real

$$E_n=(\nabla S)^2/2M+V(\boldsymbol{r})-\hbar^2\nabla^2\psi_n/2M\psi_n. \tag{2b}$$

A comparison of (2b) with Schrödinger's time independent equation reveals that $\nabla S=0$; that is, the particle is stationary with the quantum-mechanical force, $-\nabla Q$, cancelling the electrostatic interaction, $-\nabla V$. However, in the case of a central force, the stationary state solution is usually written in the form

$$\psi_{nlm}(\boldsymbol{r})=N_{nl}R_{nl}(r)P_l^{|m|}(\theta)\exp(\mathrm{i}m\phi) \tag{3a}$$

so that

$$S=m\hbar\phi-E_{nl}t \tag{3b}$$

and

$$L_z = (\boldsymbol{r} \times \nabla S)_z = m\hbar \tag{3c}$$

(Bohm 1952). If $m \neq 0$, the particle moves in a circular orbit with an angular frequency

$$\omega = m\hbar/(Mr^2 \sin^2 \theta). \tag{3d}$$

We are thus confronted with the historic problem of a charged particle supposedly undergoing acceleration without radiating; an obvious objection to the pilot wave interpretation which, surprisingly, does not seem to have been discussed in the literature.

As one possible solution to this problem we suggest that the eigenfunctions, $\psi_n(\boldsymbol{r})$, be limited to real functions. (Modifications of this hypothesis to account for magnetic interactions will be discussed in a future paper.) Thus, in the case of a central force, we may choose

$$\psi_{nlm}(\boldsymbol{r}) = N_{nl}R_{nl}P_l^{|m|}(\theta) \begin{matrix} \cos m\phi & \quad (m<0) \\ \sin m\phi & \quad (m>0) \end{matrix} \tag{3e}$$

instead of (3a).

To test the stability of these solutions, we consider the effect of a small electric field,

$$\boldsymbol{E} = \hat{k}E_0 \sin \omega t,$$

applied at time $t=0$ upon a particle in the state n. For the sake of simplicity, we shall indicate only the first quantum number, and also assume that $\omega \simeq \omega(n', n) = (E^0_{n'} - E^0_n)/\hbar$, so that, to first order in E_0,

$$\Psi(\boldsymbol{r}, t) \simeq \psi^0_n(\boldsymbol{r}) \exp(-\mathrm{i}E^0_n t/\hbar) + E_0 C(t)\psi^0_{n'}(\boldsymbol{r}) \exp(-\mathrm{i}E^0_{n'}t/\hbar) \tag{4a}$$

where

$$\mathrm{i}\hbar\dot{C} = H'_{n'n} \sin \omega t \exp[\mathrm{i}\omega(n', n)t] \tag{4b}$$

and

$$H'_{n'n} = -\int \psi^0_{n'}(\boldsymbol{r})(er\cos\theta)\psi^0_n(\boldsymbol{r})\, \mathrm{d}^3r. \tag{4c}$$

As will be shown below, the motion of the particle is now that of a forced, underdamped, harmonic oscillator and radiation should, therefore, be emitted. At velocities $|\dot{\boldsymbol{r}}| \ll c$, the radiation reaction (Plass 1961) is given by

$$\boldsymbol{F}_r = (2e^2/12\pi\varepsilon_0 Mc^3)\ddot{\boldsymbol{P}} = \tau\, \mathrm{d}^2(\nabla S)/\mathrm{d}t^2 \tag{5a}$$

where the total derivative

$$\mathrm{d}/\mathrm{d}t = \partial/\partial t + \boldsymbol{v} \cdot \nabla = \partial/\partial t + M^{-1}\nabla S \cdot \nabla. \tag{5b}$$

As can be seen with the aid of (10a) below

$$\mathrm{d}^2(\nabla S)/\mathrm{d}t^2 \simeq \partial^2(\nabla S)/\partial t^2 = \nabla(\partial^2 S/\partial t^2) \tag{5c}$$

to first order in E_0. Therefore, to the same order,

$$\boldsymbol{F}_r \simeq -\nabla\mathcal{R} \tag{6a}$$

where the 'radiation reaction potential'

$$\mathcal{R} = -\tau\, \partial^2 S/\partial t^2 = \tfrac{1}{2}\mathrm{i}\hbar\tau\, \partial^2 \ln(\Psi/\Psi^*)/\partial t^2. \tag{6b}$$

To take into account the radiation reaction, we can now modify (1*a*) to give

$$-\partial S/\partial t = (\nabla S)^2/2M + V(\boldsymbol{r}) - \hbar^2\nabla^2 R/2MR - \tau\, \partial^2 S/\partial t^2 \tag{7}$$

which together with (1*b*) is equivalent to the nonlinear Schrödinger equation:

$$i\hbar\, \partial\Psi/\partial t = [-\hbar^2\nabla^2/2M + V(\boldsymbol{r}) + E_0 H' \sin \omega t + \tfrac{1}{2} i\hbar\tau\, \partial^2 \ln(\Psi/\Psi^*)/\partial t^2]\Psi. \tag{8}$$

We see that when the particle is in a stationary state described by a real function, $\psi_n(\boldsymbol{r})$, (8) reduces to the usual Schrödinger equation.

We shall now find an approximate expression for R to first order in E_0. From (4*a*), we have

$$\frac{\Psi}{\Psi^*} \simeq \exp(-2iE_n^0 t/\hbar) \frac{1 + E_0 C(\psi_{n'}^0/\psi_n^0) \exp[-i\omega(n', n)t]}{1 + E_0 C^*(\psi_{n'}^0/\psi_n^0) \exp[i\omega(n', n)t]}. \tag{9a}$$

It is probable that the position, $\boldsymbol{r}$, of the particle is such that $\psi_n^0(\boldsymbol{r})$ is close to its maximum value so that usually $\psi_{n'}^0(\boldsymbol{r})/\psi_n^0(\boldsymbol{r}) \lesssim 1$. At any rate, since there is zero probability that $\psi_n^0(\boldsymbol{r}) = 0$ (Bohm 1952), we can always make the approximation that, for sufficiently small values of E_0,

$$\Psi/\Psi^* = \exp(-2iE_n^0 t/\hbar)(1 + 2iE_0\beta\psi_{n'}^0/\psi_n^0) \tag{9b}$$

where

$$\beta = \mathrm{Im}\{C \exp[-i\omega(n', n)r]\} = \mathrm{Im}\, C \cos[\omega(n', n)t] - \mathrm{Re}\, C \sin[\omega(n', n)t]. \tag{9c}$$

After expanding $\ln(\Psi/\Psi^*)$ to first order in E_0, we find

$$S = -E_n^0 t + \beta\hbar E_0 \psi_{n'}^0/\psi_n^0 \tag{10a}$$

$$\boldsymbol{p} = \beta\hbar E_0 \nabla(\psi_{n'}^0/\psi_n^0) \tag{10b}$$

$$\mathscr{R} = -\tau\ddot{\beta}\hbar E_0 \psi_{n'}^0/\psi_n^0. \tag{10c}$$

If we now substitute (10*c*) and (4*a*) in (8), we obtain

$$i\hbar\dot{C} = (H'_{n'n} \sin \omega t - \tau\ddot{\beta}\hbar) \exp[i\omega(n', n)t] \tag{11a}$$

instead of the usual expression, (4*b*). Separating the real and imaginary parts gives us

$$-\hbar\, \mathrm{Im}\, \dot{C} = (H'_{n'n} \sin \omega t - \tau\ddot{\beta}\hbar) \cos[\omega(n', n)t] \tag{11b}$$

$$\hbar\, \mathrm{Re}\, \dot{C} = (H'_{n'n} \sin \omega t - \tau\ddot{\beta}\hbar) \sin[\omega(n', n)t]. \tag{11c}$$

It follows that

$$\mathrm{Im}\, \dot{C} \sin[\omega(n', n)t] + \mathrm{Re}\, \dot{C} \cos[\omega(n', n)t] = 0. \tag{11d}$$

We now introduce the function

$$\alpha = \mathrm{Re}\{C \exp[-i\omega(n', n)t]\} = \mathrm{Re}\, C \cos[\omega(n', n)t] + \mathrm{Im}\, C \sin[\omega(n', n)t]. \tag{12a}$$

With the help of (11*d*)

$$\dot{\alpha} = \omega(n', n)\{\mathrm{Im}\, C \cos[\omega(n', n)t] - \mathrm{Re}\, C \sin[\omega(n', n)t]\} = \omega(n', n)\beta \tag{12b}$$

from which

$$\ddot{\beta} = \ddot{\alpha}/\omega(n', n) \tag{12c}$$

and

$$\ddot{\alpha} = \omega(n', n)\{\mathrm{Im}\, \dot{C} \cos[\omega(n', n)t] - \mathrm{Re}\, \dot{C} \sin[\omega(n', n)t]\} - \omega^2(n', n)\alpha. \tag{12d}$$

Combining (12*c*), (12*d*), (11*b*) and (11*c*) results in

$$-\tau\dddot{\alpha} + \ddot{\alpha} + \omega^2(n', n)\alpha = -[H'_{n'n}\omega(n', n)/\hbar] \sin \omega t. \tag{13}$$

The non-divergent solution of (13) which satisfies the boundary conditions, $C = 0$ and $\boldsymbol{p} = \nabla S = 0$, is

$$\alpha = \exp(-\tfrac{1}{2}\tau\omega_1^2 t)\frac{H'_{n'n}\omega(n', n)}{\hbar}\left(\frac{\omega[\omega^2(n', n) - \omega^2] - \frac{1}{2}\tau\omega_1^2\omega^3}{\omega_2\{[\omega^2(n', n) - \omega^2]^2 + \tau^2\omega^6\}} \sin \omega_2 t\right.$$
$$\left. - \frac{\tau\omega^3}{[\omega^2(n', n) - \omega^2]^2 + \tau^2\omega^6} \cos \omega_2 t\right)$$
$$+ \frac{H'_{n'n}\omega(n', n)}{\hbar} \frac{\tau\omega^3 \cos \omega t - [\omega^2(n', n) - \omega^2] \sin \omega t}{[\omega^2(n', n) - \omega^2]^2 + \tau^2\omega^6} \tag{14a}$$

where

$$\omega_1 = \omega(n', n)[1 - 2\tau^2\omega^2(n', n)] \tag{14b}$$

$$\omega_2 = \omega(n', n)[1 - (\tfrac{5}{8})\tau^2\omega^2(n', n)]. \tag{14c}$$

If the particle is an electron and $\omega(n', n) \sim 10^{16}$ (ultraviolet), ω_1 and ω_2 differ from $\omega(n', n)$ by less than one part in 10^{14}.

From (14*a*) and (12*b*), we have

$$\beta = \exp(-\tfrac{1}{2}\tau\omega_1^2 t)\frac{H'_{n'n}}{\hbar}\left(\frac{\omega_2\tau\omega^3 - \frac{1}{2}(\tau\omega_1^2/\omega_2)\{\omega[\omega^2(n', n) - \omega^2] - \frac{1}{2}\tau^2\omega_1^2\omega^2\}}{[\omega^2(n', n) - \omega^2]^2 + \tau^2\omega^6} \sin \omega_2 t\right.$$
$$\left. + \frac{\omega[\omega^2(n', n) - \omega^2]}{[\omega^2(n', n) - \omega^2]^2 + \tau^2\omega^6} \cos \omega_2 t\right)$$
$$- \frac{\omega H'_{n'n}}{\hbar} \frac{\tau\omega^3 \sin \omega t + [\omega^2(n', n) - \omega^2] \cos \omega t}{[\omega^2(n', n) - \omega^2]^2 + \tau^2\omega^6}. \tag{15}$$

From (15) and (10*b*) it is seen that, to the first order in E_0, the motion of the particle is that of a forced, underdamped harmonic oscillator. If we wish to use pictorial analogies we can think of the particle as being embedded in a highly elastic 'jelly' which resonates at the Bohr frequencies.

So far these results are encouraging. However, we still have to discover whether a more general solution and a more accurate Hamiltonian will explain such phenomena as 'spontaneous' radiation, 'transitions', etc. A more difficult problem would seem to be the explanation of magnetic interactions between stationary charged particles. A possible solution of this second problem, based on the consideration of interactions via the quantum field, will be discussed in a future paper.

We wish to emphasise that we do not claim that the model discussed in this paper is correct, even to a first approximation. However, we are convinced that it is a plausible extension of the pilot wave interpretation that needs to be analysed, without excluding the possibility of some other, more radical, modification. Finally, we wish to mention that Andrade e Silva *et al* (1960) have already claimed, on more general grounds, that atomic processes must be governed by nonlinear differential equations such as (8).

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